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Product Effect Algebras

Anatolij Dvurečenskij[1]

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We introduce a product on an effect algebra. We prove that every product effect algebra with the Riesz decomposition property (RDP), is an interval in an Abelian unital interpolation po-ring, and we show that the category of product effect algebras with the RDP is categorically equivalent with the category of unital Abelian interpolation po-rings. In addition, we show that every product effect algebra with the RDP and with 1 as a product unity is a subdirect product of antilattice product effect algebras with the RDP.

KEY WORDS: effect algebra; the Riesz decomposition property; product effect algebra; po-ring; perfect effect algebra; categorical equivalence.

1. INTRODUCTION

Effect algebras entered mathematics in 1994 because of Foulis and Bennett (1994) as partial algebras with a partially defined addition $+$. They are additive counterparts to D-posets introduced by Kôpka and Chovanec (1994), where the subtraction of comparable elements is a primary notion. They met interest of mathematicians and physicists while they give a common base for algebraic as well as fuzzy set properties of the system $\mathcal{E}(H)$ of all effects of a Hilbert space H, i.e., of all Hermitian operators A on H such that $O \leq A \leq I$, where O and I are the null and the identity operators on H. In many cases, effect algebras are intervals in unital po-groups, e.g., $\mathcal{E}(H)$ is the interval in the po-group $\mathcal{B}(H)$ of all Hermitian operators on H; this group is of great importance for physics.

We recall that every MV-algebra of Chang (1958) can also be understood as an effect algebra. For example, if M is a maximal set of mutually commuting effects on a separable Hilbert space H, then M can be converted into an MV-algebra: There is a system of Borel measurable functions $f_A : [0, 1] \to [0, 1]$, and a fixed effect operator $A_0 \in M$ such that $A = f_A(A_0)$. The MV-operations on M are then defined by $A \oplus B = (\max\{f_A + f_B, 1\})(A_0)$. $A' = I - A$, and moreover, we can

[1] Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, SK-814 73 Bratislava, Slovakia; e-mail: dvurecen@mat.savba.sk.

define in a natural way the product $A \cdot B$ of A and B via $A \cdot B := (f_A \cdot f_B)(A)$, $A, B \in M$, where $f_A \cdot f_B$ is a usual product of functions f_A and f_B.

Inspired by that, in the present note we introduce product effect algebras. We use similar notions for MV-algebras that were studied in Dvurečenskij and Riečan (1999) and Di Nola and Dvurečenskij (2001). We show that every product effect algebra with the Riesz decomposition property (RDP) is isomorphic with the interval in an Abelian unital interpolation po-ring (Section 2). We give some examples of products, including the product on perfect effect algebras (Section 3). In Section 4, we show that the category of product effect algebras with the RDP is categorically equivalent with the category of Abelian unital interpolation po-groups. Finally, we show that every product effect algebra with the RDP and with 1 as a product unity is a subdirect product of antilattice product effect algebras with the RDP (Section 5).

2. PRODUCT EFFECT ALGEBRAS

An *effect algebra* is a partial algebra $E = (E; +, 0, 1)$ with a partially defined operation $+$ and two constant elements 0 and 1 such that, for all $a, b, c \in E$,

(i) $a + b$ is defined in E iff $b + a$ is defined, and in such the case $a + b = b + a$;
(ii) $a + b$, $(a + b) + c$ are defined iff $b + c$ and $a + (b + c)$ are defined, and in such case $(a + b) + c = a + (b + c)$;
(iii) for any $a \in E$, there exists a unique element $a' \in E$ such that $a + a' = 1$; and
(iv) if $a + 1$ is defined in E, then $a = 0$.

If we define $a \leq b$ iff there exists an element $c \in E$ such that $a + c = b$, then $\leq$ is a partial ordering, and we write $c := b - a$.

Let a be any element of an effect algebra E and n an integer ($n \geq 0$). We define recurrently

$$0a := 0, \qquad 1a = a, \qquad (n+1)a = na + a, \quad n \geq 1,$$

supposing that na and $na + a$ are defined in E.

For example, if (G, u) is an Abelian unital po-group with a strong unit u,[2] and if $\Gamma(G, u) := \{g \in G : 0 \leq g \leq u\}$ is endowed with the restriction of the group addition $+$, then $(\Gamma(G, u); +, 0, u)$ is an effect algebra. More about effect algebras can be found in Dvurečenskij and Pulmannová (2000).

Let E and F be two effect algebras. A mapping $h : E \to F$ is said to be a *homomorphism* if (i) $h(a + b) = h(a) + h(b)$ whenever $a + b$ is defined in E,

[2] An element $u \in G^+$ is said to be a *strong unit* for a po-group G, if given an element $g \in G$, there is an integer $n \geq 1$ such that $-nu \leq g \leq nu$.

and (ii) $h(1) = 1$. A bijective homomorphism h such that h^{-1} is homomorphism is said to be an *isomorphism* of E and F.

A *product* on an effect algebra $E = (E; +, 0, 1)$ is any total binary operation $\cdot$ on E such that, for all $a, b, c \in E$, the following holds: If $a + b$ is defined in E, then $a \cdot c + b \cdot c$ and $c \cdot a + c \cdot b$ exist in E and

$$(a + b) \cdot c = a \cdot c + b \cdot c,$$

$$c \cdot (a + b) = c \cdot a + c \cdot b,$$

and we say that E with a product $\cdot$ is a *product effect algebra*, and we write $E = (E; +, \cdot, 0, 1)$. An element u of a product effect algebra E is said to be a *unity*, if $a \cdot u = u \cdot a = a$ for any $a \in E$.

A product $\cdot$ on E is

(i) *associative* if $(a \cdot b) \cdot c = a \cdot (b \cdot c)$, $a, b, c \in E$;
(ii) *commutative* if $a \cdot b = b \cdot a$, $a, b \in E$.

It is worth saying that if $\cdot$ is a product on E, then

(iii) $a \cdot 0 = 0 = 0 \cdot a$;
(iv) if $a \leq b$, then for any $c \in E$, $a \cdot c \leq b \cdot c$ and $c \cdot a \leq c \cdot b$.

Property (iii) follows easily from the following: $a \cdot 0 = a \cdot (0 + 0) = a \cdot 0 + a \cdot 0$, and the cancellation property gives $a \cdot 0 = 0$. Similarly, $0 \cdot a = 0$.

We recall that every effect algebra E possesses at least one commutative and associative product, namely the *zero* product, i.e. $a \cdot b = 0$ for all $a, b \in E$.

We recall that a *po-ring* is a ring $(R; +, \cdot, 0)$ such that (i) $(R; +, 0)$ is an additive Abelian po-group, and $c \cdot a \geq 0$ and $a \cdot a \geq 0$ for any $c, a \geq 0$. If u is a strong unit for R, i.e., for any $a \in R$ there is an integer $n \geq 1$ such that $a \leq nu$, and if $u \cdot u \leq u$, then the effect algebra

$$E = \Gamma(R, u)$$

is a product effect algebra with the product $\cdot$, which is the restriction of the ring product $\cdot$ to $E \times E$; the product $\cdot$ is commutative or associative on $\Gamma(R, u)$ whenever so is $\cdot$ on R.

For example, if $(\mathbb{R}; +, \cdot, 0)$ is the ring of the real numbers, then the standard interval $[0, 1] := \Gamma(\mathbb{R}, 1)$ is a product effect algebra.

We say that an effect algebra E satisfies (i) the *Riesz interpolation property* (RIP), if, for all x_1, x_2, y_1, y_2 in E, $x_i \leq y_j$ for all i, j implies there exists an element $z \in E$ such that $x_i \leq z \leq y_j$ for all i, j; (ii) the *Riesz decomposition property* (RDP), if $x \leq y_1 + y_2$ implies that there exist two elements x_1, x_2 with $x_1 \leq y_1$ and $x_2 \leq y_2$ such that $x = x_1 + x_2$.

We recall that (1) if E is a lattice, then E has trivially the RIP; the converse is not true as we see below. (2) E has the RDP iff (Dvurečenskij and

Pulmannová, 2000, Lemma 1.7.5), $x_1 + x_2 = y_1 + y_2$ implies there exist four elements $c_{11}, c_{12}, c_{21}, c_{22} \in E$ such that $x_1 = c_{11} + c_{12}, x_2 = c_{21} + c_{22}, y_1 = c_{11} + c_{21}$, and $y_2 = c_{12} + c_{22}$. (3) the RDP implies the RIP, but the converse is not true (e.g. if $E = L(H)$, the system of all closed subspaces of a Hilbert space H, then E is a complete lattice but without the RDP). On the other hand, every finite poset with the RIP is a lattice.

We recall that a poset $(E; \leq)$ is an *antilattice* if only comparable elements of E have a supremum (infimum). It is clear that any linearly ordered poset is an antilattice and every finite effect algebra with the RIP is a lattice.

There exists an effect algebra with RIP which is not a lattice:

Example 2.1. Let G be the additive group $\mathbb{R}^2$ with the positive cone of all (x, y) such that either $x = y = 0$ or $x > 0$ and $y > 0$. Then $u = (1, 1)$ is a strong unit for G. The effect algebra $E = \Gamma(G, u)$ is an antilattice having the RIP and the RDP but E is not a lattice. Moreover, if we define $(x_1, y_1) \cdot (x_2, y_2) := (x_1 \cdot x_2, y_1 \cdot y_2)$, for $(x_1, y_1), (x_2, y_2) \in E$, then $\cdot$ is a commutative and associative product on E and with unity u.

A partially ordered Abelian group $(G; +, 0)$ is said to satisfy the RDP provided, given x, y_1, y_2 in G^+ such that $x \leq y_1 + y_2$, there exist x_1, x_2 in G^+ such that $x = x^1 + x_2$ and $x_j \leq y_j$ for each j. This condition is equivalent to that by Goodearl (1986, proposition 2.1) with the following two equivalent conditions:

(a) Given x_1, x_2, y_1, y_2 in G such that $x_i \leq y_j$ for all i, j, there exists z in G such that $x_i \leq z \leq y_j$ for all i, j.
(b) Given x_1, x_2, y_1, y_2 in G^+ such that $x_1 + x_2 = y_1 + y_2$, there exist z_{11}, z_{12}, z_{21}, z_{22} in G^+ such that $x_i = z_{i1} + z_{i2}$ for each i and $y_j = z_{1j} + z_{2j}$ for each j.

According to Goodearl (1986), a group G with the RDP is said to be the *interpolation group*.

It is clear that if (G, u) is a unital interpolation group, then $E = \Gamma(G, u)$ is with the RDP.

We recall that by a *universal group* for an effect algebra E we mean a pair (G, γ) consisting of an additive Abelian group G and a G-valued measure $\gamma : E \to G$ (i.e., $\gamma(a + b) = \gamma(a) + \gamma(b)$ whenever $a + b$ is defined in E) such that the following conditions hold: (i) $\gamma(E)$ generates G. (ii) If H is an additive Abelian group and $\phi : E \to H$ is an H-valued measure, then there is a group homomorphism $\phi^* : G \to H$ such that $\phi = \phi^* \circ \gamma$. According to Foulis and Bennett (1994), every effect algebra possesses a universal group.

Ravindran (1996) (Dvurečenskij and Pulmannová, 2000, Theorem 1.17.17) proved the following important result.

Theorem 2.2. *Let E be an effect algebra with the* RDP. *Then there exists a unital interpolation group (G, u) with a strong unit u such that $\Gamma(G, u)$ is isomorphic with E, and there is a G-valued injective measure γ such that (G, γ) is a universal group for E.*

In Dvurečenskij (Theorem 5.8) we have proved that the category of effect algebras with the RDP is categorically equivalent to the category of unital Abelian interpolation po-groups: Let $\mathcal{EA}_{\mathcal{RDP}}$ be the category of effect algebras with the RDP whose objects are effect algebras and morphisms are homomorphisms of effect algebras, and let $\mathcal{UIG}$ be the category of unital interpolation po-groups (G, u) with a fixed strong unit u and whose morphisms are homomorphisms of unital po-groups, i.e., positive homomorphisms of unital po-groups that preserve fixed strong units.

Theorem 2.3. *The mapping $\Gamma : \mathcal{UIG} \to \mathcal{EA}_{\mathcal{RDP}}$ defines the categorical equivalence of the category $\mathcal{UIG}$ of unital interpolation po-groups and the category of effect algebras with the* RDP.

In addition, suppose that $h : \Gamma(G, u) \to \Gamma(H, v)$ is a homomorphism of effect algebras with the RDP, *then there is a unique homomorphism $f : (G, u) \to (H, v)$ of unital po-groups such that $h = \Gamma(f)$, and*

(i) *if h is surjective, so is f;*
(ii) *if h is injective, so is f.*

We now present a crucial result for our investigation.

Theorem 2.4. *Let $(E; +, \cdot, 0, 1)$ be a product effect algebra with the* RDP. *Then there exists a unique (up to isomorphism) unital po-ring (R, u) satisfying the* RDP *with the product. and with $u \cdot u \le u$ such that $E \cong \Gamma(R, u)$ and $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$. where ϕ is an isomorphism of E onto $\Gamma(R, u)$ preserving the product.*

If $\cdot$ is commutative or associative, so is $\cdot$ on R.

Proof: Let E be an effect algebra with a product. According to Theorem 2.2, there is a unital group (R, u) with a strong unit u satisfying the RDP, and an isomorphism ϕ from E onto $\Gamma(R, u)$. We can define the product $\cdot$ on $\Gamma(R, u)$ as follows

$$\phi(a) \cdot \phi(b) := \phi(a \cdot b), \quad a, b \in E.$$

Because $\Gamma(R, u)$ generates the positive cone R^+ of R, ϕ preserves all existing $+$ in R, and we see that $\cdot$ is a product on $\Gamma(R, u)$.

Given $g \in R^+$ there exist $a_1, \ldots, a_n \in E$ such that $g = \sum_{i=1}^n \phi_{(ai)}$. For any $\phi(c)$, where $c \in E$, we define

$$g \cdot \phi(c) = \phi(a_1 \cdot c) + \cdots + \phi(a_n \cdot c). \tag{2.1}$$

We claim that (2.1) is defined unambiguously. Indeed, if $g = \sum_{j=1}^m \phi(b_j)$, for some $b_1, \ldots, b_m \in E$, because of the RDP, there exist elements $c_{ij} \in E$ such that $a_i = \sum_{j=1}^m c_{ij}$ and $b_j = \sum_{i=1}^n c_{ij}$ for all i, $1 \leq i \leq n$, and all j, $1 \leq j \leq m$. Then

$$\begin{aligned}\sum_{i=1}^n \phi(a_i \cdot c) &= \sum_{i=1}^n \phi\left(\left(\sum_{j=1}^m c_{ij}\right) \cdot c\right) = \sum_{i=1}^n \phi\left(\sum_{j=1}^m (c_{ij} \cdot c)\right)\\ &= \sum_{i=1}^n \sum_{j=1}^m \phi(c_{ij} \cdot c) = \sum_{j=1}^m \sum_{i=1}^n \phi(c_{ij} \cdot c)\\ &= \sum_{j=1}^m \phi\left(\sum_{i=1}^n c_{ij} \cdot c\right) = \sum_{j=1}^m \phi\left(\left(\sum_{i=1}^n c_{ij}\right) \cdot c\right)\\ &= \sum_{j=1}^m \phi(b_j \cdot c),\end{aligned}$$

which proves that $\cdot$ on $R^+ \times \Gamma(R, u)$ is correct. We now extend $\cdot$ to $R \times \Gamma(R, u)$ as follows: If $g = g_1 - g_2$, $g_1, g_2 \in R^+$, then

$$g \cdot \phi(c) := g_1 \cdot \phi(c) - g_2 \cdot \phi(c).$$

Since if $g_1 - g_2 = h_1 - h_2$ for $g_i, h_i \in R^+$, $i = 1, 2$, then $g_1 + h_2 = h_1 + g_2$, by (2.1) we have

$$\begin{aligned}(g_1 + h_2) \cdot \phi(c) &= (h_1 + g_2) \cdot \phi(c),\\ g_1 \cdot \phi(c) + h_2 \cdot \phi(c) &= h_1 \cdot \phi(c) + g_2 \cdot \phi(c),\\ g_1 \cdot \phi(c) - g_2 \cdot \phi(c) &= h_1 \cdot \phi(c) + h_2 \cdot \phi(c).\end{aligned}$$

Now let $c \in R^+$. Then $c = \phi(c_1) + \ldots + \phi(c_s)$, where $c_t \in M$, $t = 1, \ldots, s$. We extend $\cdot$ to $R \times R^+$ as follows

$$g \cdot c := \sum_{t=1}^s g \cdot \phi(c_t), \quad g \in G.$$

If $c = \sum_{w=1}^{\nu} \phi(dw)$, using the RDP, we have as above

$$\sum_{t=1}^{s} g \cdot \phi(c_t) = \sum_{w=1}^{\nu} g \cdot \phi(d_w).$$

It is clear that the "multiplication" $\cdot$ can be extended to the whole $R \times R$: If $c = c_1 - c_2$, where $c_1, c_2 \in R^+$, let

$$g \cdot c := g \cdot c_1 - g \cdot c_2.$$

It is evident that if $g, h \in R^+$, then $g \cdot h \in R^+$, and $u \cdot u \leq u$, and because of property (ii), $\cdot$ is associative on R, so that $(R; +, \cdot, 0)$ is a po-ring with a strong unit u, which proves the theorem.

The commutativity or associativity of the product on the ring R follows from its construction. □

We present now some effect algebras that admit products.

Proposition 2.5. *If e is unity of a product effect algebra E, then $e = 1$. Moreover, 1 is unity for $\cdot$ if and only if, for any $a \in E$, $a \cdot 1 \geq a$ and $1 \cdot a \geq a$.*

Proof: Let e be unity for the product. Then $1 \cdot 1 \leq 1 = 1 \cdot e \leq 1 \cdot 1$, which proves $1 \cdot 1 = 1$. Let $a \in E$. Then $a \cdot 1 \geq a \cdot e = a$ and $1 \cdot a \geq e \cdot a = a$. In addition, $a + a' = 1 = 1 \cdot 1 = (a + a') \cdot 1 = a \cdot 1 + a' \cdot 1 \geq a + a'$, which entails $a \cdot 1 = a$. By symmetry, we have $1 \cdot a = a$ for any $a \in E$, and in addition $1 = 1 \cdot e = e$.

Now assume that for any $a \in E$, $a \cdot 1 \geq a$ and $1 \cdot a \geq a$. Then $1 = a + a' \leq a \cdot 1 + a' \cdot 1 = (a + a') \cdot 1 = 1 \cdot 1 \leq 1$, which proves that 1 is unity for $\cdot$ on E. □

Proposition 2.6. *A finite effect algebra E, which is a lattice and with the* RDP, *admits a product $\cdot$ such that $a \cdot 1 = a = 1 \cdot a$ for any $a \in E$ if and only if E is a Boolean algebra, i.e., $a \vee a' = 1$ for any $a \in E$. If this is the case, then $a \cdot b = a \wedge b \in E$.*

Proof: Every lattice effect algebra with the RDP can be converted into an MV-algebra. Therefore, the assertion follows from Dvurečenskij and Pulmannová (2000, Theorem 5.3.17). □

3. PRODUCT ON PERFECT EFFECT ALGEBRAS

In the present section, we show that we can even define a product on perfect effect algebras. Such effect algebras were introduced and studied in Dvurečenskij.

Let G be a directed Abelian po-group and define the lexicographical product

$$G(\mathbb{Z}) := \mathbb{Z} \times_{\text{lex}} G, \tag{3.1}$$

where $\mathbb{Z}$ is the group of all integers. Then the element (1, 0) is a strong unit in the po-group $G(\mathbb{Z})$ and

$$E(G) := \Gamma(G(\mathbb{Z}), (1, 0)) \tag{3.2}$$

is an effect algebra. Every element $a \in E(G)$ is of the form either $a = (1, -g)$ or $a = (0, g)$, where $g \in G^+$. In addition, if G is a directed interpolation group, then $G(\mathbb{Z})$ is an interpolation group (Goodearl, 1986, corrolary 2.12) and $E(G)$ satisfies the RIP.

An *ideal* of an effect algebra E is a nonempty subset I of E such that (i) $x \in E, y \in I, x \leq y$ imply $x \in I$, and (ii) if it $x, y \in I$ and $x + y$ is defined in E, then $x + y \in I$. An ideal I is said to be the *Riesz ideal* if $x \in I$, $a, b \in E$, and $x \leq a + b$, there exist $a_1, b_1 \in I$ such that $x \leq a_1 + b_1$ and $a_1 \leq a$ and $b_1 \leq b$.

For example, if E is with the RDP, then any ideal of E is Riesz.

A proper ideal I of an effect algebra E is said to be *maximal* if it is not a proper subset of another proper ideal of E. By Zorn's lemma E possesses at least one maximal ideal, and let $\mathcal{M}(E)$ be the set of all maximal ideals of E. We define the *radical* of E, Rad(E), via

$$\text{Rad}(E) = \cap\{I \ : \ I \in \mathcal{M}(E)\}.$$

An element a is said to be *infinitesimal* if na is defined in E for any integer $n \geq 1$, and denote by Infinit(E) the set of all infinitesimals of E. Then (i) $0 \in \text{Infinit}(E)$, (ii) if $b \in E$, $a \in \text{Infinit}(E)$, and $b \leq a$, then $b \in \text{Infinit}(E)$, and (iii) $1 \notin$ Infinit(E). By Proposition 4.1 of Dvurečenskij if E is an effect algebra satisfying the RDP, then

$$\text{Infinit}(E) \subseteq \text{Rad}(E). \tag{3.3}$$

Since it can happen that in (3.3) we have the proper inclusion, according to Dvurečenskij, an effect algebra E with the RDP is said to have the *Rad-property* if in (3.3) we have the equality. In Dvurečenskij, we have introduced perfect effect algebras: We say that an effect algebra E with the Rad-property is *perfect* if, for any element $a \in E$, either $a \in \text{Rad}(E)$ or $a' \in \text{Rad}(E)$. According to Dvurečenskij, (Proposition 5.3), every perfect effect algebra E is of the form (3.2) for some directed Abelian interpolation po-group G.

We say that G^+ of a po-group G is *Archimedean* if, for some $a, b \in G^+$ with $na \leq b$ for any $n \geq 1$, $a = 0$.

Proposition 3.1. *Let E be a perfect effect algebra and define a binary operation. as follows. For $a, b \in \mathrm{Rad}(E)$, let*

$$\begin{aligned} a \cdot b &= 0, \\ a \cdot b' &= a, \\ a' \cdot b &= b, \\ a' \cdot b' &= (a + b)'. \end{aligned} \tag{3.4}$$

Then $\cdot$ is a commutative product on E such that $a \cdot 1 = a = 1 \cdot a$ for any $a \in E$.

Let $E = \Gamma(\mathbb{Z} \times_{\mathrm{lex}} G, (1, 0))$ and let G^+ be Archimedean. Then on E there is a unique product $\cdot$ such that $a \cdot 1 = a = 1 \cdot a$ for any $a \in E$; this product is defined by (3.4).

Proof: Since $\mathrm{Rad}(E) \cap \mathrm{Rad}'(E) = \emptyset$, we see that $\cdot$ is correctly defined by (3.4), and it is easy to verify that it is a product in question on E.

Let a and b be two element of $\mathrm{Rad}(E)$. Then nb is defined in E for any integer $n \geq 1$, and by Dvurečenskij, (Proposition 5.1), we have

$$a = a \cdot 1 = a \cdot ((nb) + (nb)') = n(a \cdot b) + a \cdot (nb)' \geq n(a \cdot b).$$

The Archimedeanicity of G^+ entails that $a \cdot b = 0$. Hence $a = a \cdot b + a \cdot b' = a.b'$. Similarly $a = (b' \cdot a) + (b \cdot a) = b' \cdot a$, and $a' = a' \cdot b' + a' \cdot b' = a' \cdot b' + b$, which gives $a' \cdot b' = (a + b)'$. □

Proposition 3.2.

Let E be an effect algebra with the RDP *satisfying the Rad-property such that* Rad*(E) is Archimedean, i.e., if $na \leq b$ for any $n \geq 1$ and for some $b \in \mathrm{Rad}(E)$, then $a = 0$. If $\cdot$ is a product on E such that $a \cdot 1 = a = 1 \cdot a$ for any $a \in E$, then, for any $a, b \in \mathrm{Rad}(E)$,* (3.4) *holds.*

Proof: The first three identities follow the same ideas as the proof of Proposition 3.1. For the last one we have

$$1 = (a + a') \cdot (b + b') = a \cdot b + a \cdot b' + a' \cdot b + a' \cdot b' = (a + b) + (a' \cdot b'),$$

which entails that $a' \cdot b' = (a + b)'$. □

Remark 3.3. It is worth recalling that if an effect algebra E satisfying the RDP and the Rad-property has an Archimedean radical, then there is a unique binary operation$\cdot$: $(\mathrm{Rad}(E) \cup \mathrm{Rad}'(E)) \times (\mathrm{Rad}(E) \cup \mathrm{Rad}'(E)) \to \mathrm{Rad}(E) \cup \mathrm{Rad}(E)'$, which is commutative and associative and for all $a, b, c \in \mathrm{Rad}(E) \cup \mathrm{Rad}(E)$ we have $(a + b) \cdot c = a \cdot c + b \cdot c$, $a \cdot 1 = a$; it is defined via (3.4). On the other hand it does not mean that $\cdot$ can be extended to a product on the whole E, satisfying $a \cdot 1 = a = 1 \cdot a$ for any $a \in E$. for example, if E is a finite lattice, then $\mathrm{Rad}(E) = \{0\}$, and $\cdot$ can be extended iff E is a Boolean algebra (Proposition 2.6). An extension of $\cdot$ is possible, e.g., if E is a perfect effect algebra with an Archimedean radical (Proposition 3.1).

4. PRODUCT EFFECT ALGEBRAS AND CATEGORICAL EQUIVALENCE

In this section, we show that the category of product effect algebras with the RDP is categorically equivalent with the category of Abelian unital interpolation po-rings.

Denote by $\mathcal{PROD}_{\mathcal{EA}}$ the category of product effect algebras, i.e., its objects are product effect algebras, and morphisms are homomorphisms of effect algebras also preserving. We denote by $\mathcal{R}$ the category of associative unital interpolation po-rings (R, u) with a distinguished strong unit u such that $u \cdot u \leq u$, and its morphisms are homomorphisms of po-groups that preserve. and the distinguished strong units. We denote by $\Gamma_{\mathcal{R}}$ a map from $\mathcal{R}$ into $\mathcal{PROD}_{\mathcal{EF}}$ defined by $\Gamma_{\mathcal{R}}((R; +, \cdot, 0, \leq, u)) := (\Gamma(R, u); +, \cdot, 0, u)$, and $\Gamma_{\mathcal{R}}(f) := f \mid \Gamma(R, u)$.

Proposition 4.1. *$\Gamma_{\mathcal{R}}$ is a faithful and full functor from $\mathcal{R}$ to $\mathcal{PROD}_{\mathcal{EA}}$.*

Proof: Let h_1 and h_2 be two morphisms from $(R; +, \cdot, 0, \leq, u)$ into $(R'; +, \cdot, 0, \leq, u')$ such that $\Gamma_{\mathcal{R}}(h_1) = \Gamma_{\mathcal{R}}(h_2)$. Then $h_1(a) = h_2(a)$ for any $a \in \Gamma(R, u)$. Since $\Gamma(R, u)$ generates R^+ and R, we have that $h_1(g) = h_2(g)$ for any $g \in R$, which proves that $\Gamma_{\mathcal{R}}$ is faithful.

To prove that $\Gamma_{\mathcal{R}}$ is a full functor, suppose that f is a morphism from $\Gamma(R, u)$ into $\Gamma(R', u')$. Since $\Gamma(R, u)$ generates R, due to the RDP, f can be uniquely extended to a group homomorphism $\hat{f}$ from R into R'.

We show that $\hat{f}$ preserves the product $\cdot$ in R.

Let $a, b \in R^+$ There exist $a_1, \ldots, a_n, b_1, \ldots, b_m \in \Gamma(R, u)$ such that $a = a_1 + \cdots + a_n$ and $b = b_1 + \cdots + b_m$. Then $a \cdot b = \sum_{i=1}^n \sum_{j=1}^m a_i \cdot b_j$ and $a_i \cdot b_j \in \Gamma(R, u)$. Calculate $\hat{f}(a \cdot b) = \sum_{i=1}^n \sum_{i=1}^m \hat{f}(a_i \cdot b_j) = \sum_{i=1}^n \sum_{j=1}^m \hat{f}(a_i) \cdot \hat{f}(b_j) = \left(\sum_{i=1}^n \hat{f}(a_i)\right) \cdot \left(\sum_{j=1}^m \hat{f}(b_j)\right) \hat{f}(a) \cdot \hat{f}(b)$.

If now $a, b \in R$, then $a = a_1 - a_2$ and $b = b_1 - b_2$, where $a_1, a_2, b_1, b_2 \in R^+$. Then $\hat{f}(a \cdot b) = \hat{f}(a_1 \cdot b_1 - a_1 \cdot b_2 - a_2 \cdot b_1 + a_2 \cdot b_2) = \hat{f}(a_1 \cdot b_1) - \hat{f}(a_1 \cdot b_2) - \hat{f}(a_2 \cdot b_1) + \hat{f}(a_2 \cdot b_2) = \hat{f}(a_1) \cdot \hat{f}(b_1) - \hat{f}(a_1) \cdot \hat{f}(b_2) - \hat{f}(a_2) \cdot \hat{f}(b_1) + \hat{f}(a_2) \cdot \hat{f}(b_2) = \hat{f}(a) \cdot \hat{f}(b)$.

Consequently, we have proved that $\hat{f}$ is a morphism from $(R; +, \cdot, 0, \leq, u)$ into $(R'; +, \cdot, 0, \leq, u')$ such that $\Gamma_{\mathcal{R}}(\hat{f}) = f$. □

We are now ready to present the main statement of the present section.

Theorem 4.2. *The functor $\Gamma_{\mathcal{R}}$ defines a categorical equivalence of the category $\mathcal{R}$ of unital associative interpolation po-rings with a strong unit u such that $u \cdot u \leq u$ and the category $\mathcal{PROD}_{\mathcal{EA}}$ of product effect algebras.*

Proof: According to MacLane (1971, Theorem IV.4.1), to prove that $\Gamma_{\mathcal{R}}$ is an equivalence of the categories in question, it is necessary and sufficient to show that $\Gamma_{\mathcal{R}}$ is faithful and full, and each object E from $\mathcal{PROD}_{\mathcal{EA}}$ is isomorphic to $\Gamma_{\mathcal{R}}((R; +, \cdot, 0, \leq, u))$ for some object $(R; +, \cdot, 0, \leq, u)$ in $\mathcal{R}$.

According to Proposition 4.1, $\Gamma_{\mathcal{R}}$ is faithful and full, and by Theorem 2.4, there exists an object $(R; +, \cdot, 0, \leq, u)$ in $\mathcal{R}$ such that $\Gamma_{\mathcal{R}}((R; +, \cdot, 0, \leq, u))$ is isomorphic to E. This proves the theorem. □

5. SUBDIRECT PRODUCT OF PRODUCT EFFECT ALGEBRAS

In the present section, we show that every product effect algebra with the RDP and with 1 as a product unity is a subdirect product of antilattice product effect algebras with the RDP. To show that, we give some results on ideals and quotient effect algebras.

Let $\cdot$ be a product on an effect algebra E. We say that an $\cdot$ ideal I of E is an $\cdot$ *ideal* of E if $a \in I$ and $b \in E$ entail $a \cdot b \in I$ and $b \cdot a \in I$. If 1 is a product unity, then every ideal of E is a $\cdot$ ideal of E.

Let P be a proper ideal of E. We define a relation $\sim_P$ on E via $a \sim_P b$ iff $a - e = b - f$ for some $e, f \in P$. According to Dvurečenskij and Pulmannová (2000, Section 3.1.2). we have that $\sim_P$ is an equivalence such that (i) $a + b \in E$, $a_1 + b_1 \in E$, $a \sim_P a_1$, $b \sim_P b_1$ imply $(a + b) \sim_P (a_1 + b_1)$, (ii) $a \sim_P b$ implies $a' \sim_P b'$, (iii) $a + b \in E$, $c \sim_P a$ imply there exists an element $d \in E$ such that $d \sim_P b$ and $d + c \in E$, (iv) $a + b, a_1 + b_1 \in E$, $a_1 \sim_P, a (a_1 + b_1) \sim_P (a + b)$ imply $b_1 \sim_P b$. If we define $a/P := [a] := [a]_P := \{b \in E : b \sim_P a\}$, then $E/P := \{[a]_P : a \in E\}$ is an effect algebra, where $[a] + [b] = [c]$ iff there exist $a_1 \in [a]$, $b_1 \in [b]$, $c_1 \in [c]$ such that $a_1 + b_1 = c_1$. As the constant elements in E/P we take [0] and [1].

We recall that

$$[a]_P \leq [b]_P \text{ in } E/P \quad \Leftrightarrow \quad \text{there exists } a_1 \in [a]_P \text{ such that } a_1 \leq b. \tag{5.1}$$

We say that an ideal P of an effect algebra E with the RDP is *prime* if, for all ideals I and J of E, $I \cap J \subseteq P$ implies $I \subseteq P$ or $J \subseteq P$. We denote by $\mathcal{P}(E)$ the set of all prime ideals of E.

Let a be a nonzero element of E. By Zorn's lemma, there exists an ideal V not containing a and which is maximal with respect to this property. Such an ideal is said to be a *value* of a in E.

According to Dvurečenskij, (i) every value is a prime ideal of E, (ii) E/P is an effect algebra with the RDP whenever P is a proper ideal of E with the RDP, and (iii) E/P is an antilattice effect algebra if P is prime.

Proposition 5.1. *Let P be a proper ideal of a product effect algebra $(E;+,\cdot,0,1)$ with the* RDP. *If $a_1 \sim_P a_2$ and $b_1 \sim_P b_2$, then $a_1 \cdot b_1 \sim_P a_2 \cdot b_2$. In addition, E/P is a product effect algebra with the induced product $\cdot$ defined via*

$$[a]_P \cdot [b]_P := [a \cdot b]_P, \quad a, b \in E, \tag{5.2}$$

and the canonical mapping $f : E \to E/P$ given by $f(a) = [a]_P$, $a \in E$, is a surjective homomorphism preserving all existing joins, meets, and the product in E.

Proof: There are $e, f, u, v \in P$ such that $a_1 - e = a_2 - f$ and $b_1 - u = b_2 - v$. Then

$$\begin{aligned}(a_1 - e) \cdot (b_1 - u) &= (a_2 - f) \cdot (b_2 - v),\\ a_1 \cdot (b_1 - u) - e \cdot (b_1 - u) &= a_2 \cdot (b_2 - v) - f \cdot (b_2 - v),\\ a_1 \cdot b_1(a_1 \cdot u + e \cdot (b_1 - u)) &= a_2 \cdot b_2 - (a_2 \cdot v + f \cdot (b_2 - v)).\end{aligned}$$

Since $x := a_1 \cdot u + e \cdot (b_1 - u) \in P$ and $y := a_2 \cdot v + f \cdot (b_2 - v) \in P$, we have $a_1 \cdot b_1 - x = a_2 \cdot b_2 - y$, i.e., $a_1 \cdot b_1 \sim_P a_2 \cdot b_2$. Therefore, the product on E/P defined by (5.2) is a product in question.

The rest is now clear; see also Dvurečenskij (Proposition 6.9). □

Let $\{E_i\}_{i \in I}$ be an indexed system of effect algebras. The Cartesian product $\prod_{i \in I} E_i$ can be organized into an effect algebra with the partial addition defined by coordinates. Each E_i is with the RDP iff so is $\prod_i E_i$. If every E_i is a product effect algebra, then so is $\prod_i E_i$, where the product is defined by coordinates.

We say that an effect algebra is a *subdirect product* of effect algebras $\{E_i\}_{i \in I}$ if there is an injective homomorphism $f : E \to \prod_{i \in I} E_i$ such that, for every $j \in I \pi_j, \circ f$ is a surjective homomorphism from E onto E_j, where π_j is the jth projection of $\prod_i E_i$ onto E_j. In an analogical manner we define a subdirect product of product effect algebras assuming that the function f preserves also the product in E, and every $\pi_j \circ f$ is a surjective homomorphism of effect algebras preserving product.

Theorem 5.2. *Every product effect algebra $(E;+,\cdot,0,1)$ with the* RDP *having* 1 *as a product unity is a subdirect product of antilattice product effect algebras with* *the* RDP. *In addition, we can find an injective homomorphism into the subdirect product that preserves all existing meets, joins, and the product in E.*

Proof: According to Dvurečenskij (Theorem 7.2), every effect algebra with the RDP is a subdirect product of antillatice effect algebras $\{E/P : P \in \mathcal{P}(E), P \neq E\}$, and each of them is with the RDP. The corresponding injective mapping $f : E \to \prod_P E/P$ is defined via $f(a) = ([a]_P)_{P \in \mathcal{P}(E)}$, $a \in E$.

According to Proposition 5.1, every E/P is a product effect algebra. In addition, the mapping f preserves all existing meets, joins (see Dvurečenskij, Theorem 7.2]), and the product in E, and $\pi_P \circ f$ is a surjective homomorphism from E onto E/P preserving. □

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